

THREE SOLUTIONS FOR A FRACTIONAL ELLIPTIC PROBLEMS WITH CRITICAL AND SUPERCRITICAL GROWTH

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ABSTRACT. In this paper, we deal with the existence and multiplicity of solutions for the fractional elliptic problems involving critical and supercritical Sobolev exponent via variational arguments. By means of the truncation combining with the Moser iteration, we prove that the problems has at least three solutions.

1. INTRODUCTION AND MAIN RESULT

2. INTRODUCTION

In this paper, we consider the existence and multiplicity of solutions for the fractional elliptic problem

$$\begin{cases} (-\Delta)^s u = \lambda f(x, u) + \mu |u|^{p-2} u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \quad (2.1)$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is a smooth bounded domain, $(-\Delta)^s$ stands for the fractional Laplacian, $p \geq 2_s^* = \frac{2N}{N-2s}$, μ and λ are nonnegative constants and $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carátheodory function.

The fractional Laplacian appears in diverse areas including physics, biological modeling, mathematical finances and especially partial differential equations involving the fractional Laplacian have been attracted by researchers. An important feature of the fractional Laplacian is its nonlocal property, which makes it difficult to handle. Recently, Caffarelli and Silvestre [1] developed a local interpretation of the fractional Laplacian through Dirichlet-Neumann maps. This is commonly used in the recent literature since it allows to write nonlocal problems in a local way and this permits to use the variational methods for these kinds of problems.

Based on these extensions, many authors studied nonlinear problem of the form $(-\Delta)^s u = f(x, u)$ for a certain function $f : \mathbb{R}^N \rightarrow \mathbb{R}$. Among others, it is worthwhile to mention the work by Cabré-Tan [2] and Tan [3] when $s = \frac{1}{2}$. They established the existence of positive solutions for equations having the subcritical growth, their regularity and symmetry properties. Recently, and for the subcritical case, Choi, Kim and Lee [4] developed a nonlocal analog of the results by Han [5] and Rey [6].

In this paper, we study the existence and multiplicity of solutions for the problem with critical and supercritical growth. For our problem, the first difficulty lies in that the fractional Laplacian operator $(-\Delta)^s$ is nonlocal, the nonlocal property of $(-\Delta)^s$ makes some calculations difficult. To overcome this difficult, we do not

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work on the space $H_0^s(\Omega)$ directly, we transform the nonlocal problem into a local problem by the extension introduced by Caffarelli and Silvestre in [1]. After this extension, the problem (2.1) can be reduced to the problem

$$\begin{cases} \operatorname{div}(y^{1-2s}\nabla w) = 0, & \text{in } \mathcal{C}, \\ w = 0, & \text{on } \partial_L \mathcal{C}, \\ \partial_\nu^s w = \lambda f(x, w) + \mu|w|^{p-2}w, & \text{in } \Omega \times \{0\}, \end{cases} \quad (2.2)$$

where ν is the outward unit normal vector to \mathcal{C} on $\Omega \times \{0\}$ and

$$\partial_\nu^s w(x, 0) := - \lim_{y \rightarrow 0^+} y^{1-2s} \frac{\partial w}{\partial y}(x, y), \quad \forall x \in \Omega.$$

Obviously, the equation (2.2) is a local problem.

The second difficult lies in that problem (2.1) is a supercritical problem. Hence, we can not use directly the variational techniques because the corresponding energy functional is not well-defined on the Sobolev space $H_0^s(\Omega)$. To overcome this difficult, one usually uses the truncation and the Moser iteration. This spirit has been widely used in the supercritical Laplacian equation in the past few decades, see [7, 8, 9, 10, 11, 13] and references therein.

The aim of this paper is to study the problem (2.1) when $p \geq 2_s^*$. During this study we develop some nonlocal techniques which also have their own interests. In order to state our main results, we formulate the following assumptions:

- (f₁) $\lim_{|t| \rightarrow +\infty} \frac{f(x, t)}{|t|} = 0$ uniformly in $x \in \Omega$;
- (f₂) $\lim_{|t| \rightarrow 0} \frac{f(x, t)}{|t|} = 0$ uniformly in $x \in \Omega$;
- (f₃) $\sup_{u \in H_0^s(\Omega)} \int_\Omega F(x, u) dx > 0$, and for every $M > 0$, $f(x, u) \in L^\infty(\Omega)$ for each $|u| \leq M$, where $F(x, u) = \int_0^u f(x, t) dt$.

Set

$$\theta := \frac{1}{2} \inf \left\{ \frac{\int_\Omega |(-\Delta)^{\frac{s}{2}} u|^2 dx}{\int_\Omega F(x, u) dx} : u \in H_0^s(\Omega), \int_\Omega F(x, u) dx > 0 \right\}.$$

The main results are as follows.

Theorem 2.1. *Assume that (f₁) – (f₃) hold. Then there exists $\delta > 0$ such that for any $\mu \in [0, \delta]$, there exist an compact interval $[a, b] \subset (\frac{1}{\theta}, +\infty)$ and a constant $\gamma > 0$ such that for each $\lambda \in [a, b]$, the problem (2.1) has at least three solutions in $H_0^s(\Omega)$, whose norms are less than γ .*

For the general problem

$$\begin{cases} (-\Delta)^s u = \lambda f(x, u) + \mu g(x, u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \quad (2.3)$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, and

$$(g) \quad |g(x, u)| \leq C(1 + |u|^{p-1}), \text{ where } p \geq 2_s^* = \frac{2N}{N-2s}, \quad C > 0.$$

If f satisfies the conditions (f₁) – (f₃), we also have the following result similar to Theorem 2.1.

Theorem 2.2. *Let f satisfy $(f_1) - (f_3)$ and g satisfy (g) . Then there exists $\delta > 0$ such that for any $\mu \in [0, \delta]$, there exist an compact interval $[a, b] \subset (\frac{1}{\theta}, +\infty)$ and a constant $\gamma > 0$ such that for each $\lambda \in [a, b]$, the problem (2.3) has at least three solutions in $H_0^s(\Omega)$, whose $H_0^s(\Omega)$ -norms are less than γ .*

The paper is organized as follows. In Section 2, we introduce a variational setting of the problem and present some preliminary results. In Section 3, some properties of the fractional operator are discussed, and apply the truncation and the Moser iteration to obtain the proof of Theorem 2.1 and Theorem 2.2.

For convenience we fix some notations. $L^p(\Omega)$ ($1 < p \leq \infty$) denotes the usual Sobolev space with norm $\|\cdot\|_{L^p}$; $C_0(\bar{\Omega})$ denotes the space of continuous real functions in $\bar{\Omega}$ vanishing on the boundary $\partial\Omega$; C or C_i ($i = 1, 2, \dots$) denote any positive constant.

3. PRELIMINARIES AND FUNCTIONAL SETTING

In this section we recall some basic properties of the fractional Laplacian. In the entire space, the operator $(-\Delta)^s$ in \mathbb{R}^N , $0 < s < 1$, is defined through Fourier transform \mathcal{F} , by

$$\mathcal{F}[(-\Delta)^s u](\xi) = |\xi|^{2s} \mathcal{F}[u](\xi).$$

On a bounded domain Ω , we define $(-\Delta)^s$ through the spectral decomposition of $-\Delta$ in $H_0^1(\Omega)$:

$$(-\Delta)^s u = \sum_{i=1}^{\infty} \mu_i^s u_i \varphi_i,$$

where $u = \sum_{i=1}^{\infty} u_i \varphi_i$, $u_i = \int_{\Omega} u \varphi_i dx$ and (μ_i, φ_i) are the eigenvalues and corresponding eigenfunctions of $-\Delta$ on $H_0^1(\Omega)$. The fractional Laplacian is well defined in the fractional Sobolev space $H_0^s(\Omega)$,

$$H_0^s(\Omega) = \{u = \sum a_j \varphi_j \in L^2(\Omega) : \|u\|_{H_0^s} = (\sum a_j^2 \lambda_j^s)^{\frac{1}{2}} < \infty\},$$

which is a Hilbert space endowed with the following inner product

$$\langle \sum_{i=1}^{\infty} a_i \varphi_i, \sum_{i=1}^{\infty} b_i \varphi_i \rangle = \sum_{i=1}^{\infty} a_i b_i \mu_i^s,$$

and we have the following expression for this inner product

$$\langle u, v \rangle = \int_{\Omega} (-\Delta)^{\frac{s}{2}} u \cdot (-\Delta)^{\frac{s}{2}} v dx = \int_{\Omega} (-\Delta)^s u v dx, \quad \forall u, v \in H_0^s(\Omega).$$

We will often work with an equivalent definition based on an appropriate extension problem introduced by Caffarelli and Silvestre [1]. Let Ω be a bounded smooth domain in \mathbb{R}^N , the half cylinder with base Ω denote by $\mathcal{C} = \Omega \times (0, \infty) \subset \mathbb{R}_+^{N+1}$ and its lateral boundary given that $\partial_L \mathcal{C} = \partial\Omega \times [0, \infty)$, where

$$\mathbb{R}_+^{N+1} = \{(x, y) = (x_1, x_2, \dots, x_n, y) \in \mathbb{R}^{N+1} : y > 0\}.$$

The space $H_{0,L}^s(\mathcal{C})$ is defined as the completion of

$$C_{0,L}^s(\mathcal{C}) = \{w \in C^\infty(\bar{\mathcal{C}}) : w = 0 \text{ on } \partial_L \mathcal{C}\}$$

with respect to the norm

$$\|w\|_{H_{0,L}^s(\mathcal{C})} = \left(\int_{\mathcal{C}} y^{1-2s} |\nabla w|^2 dx dy \right)^{\frac{1}{2}}.$$

This is a Hilbert space endowed with the following inner product

$$\langle w, v \rangle = \int_{\mathcal{C}} y^{1-2s} \nabla w \nabla v dx dy, \quad \forall w, v \in H_{0,L}^s(\mathcal{C}).$$

Definition 3.1. We say that $u \in H_0^s(\Omega)$ is a solution of Equation (2.1) such that for every function $\varphi \in H_0^s(\Omega)$, it holds

$$\int_{\Omega} (-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} \varphi dx = \lambda \int_{\Omega} f(x, u) \varphi dx + \mu \int_{\Omega} |u|^{p-2} u \varphi dx.$$

Associated with problem (2.1) we consider the energy functional

$$I(u) = \frac{1}{2} \int_{\Omega} |(-\Delta)^{\frac{s}{2}} u|^2 dx - \lambda \int_{\Omega} F(x, u) dx - \frac{\mu}{p} \int_{\Omega} |u|^p dx.$$

We now conclude the main ingredients of a recently developed technique which can deal with fractional power of the Laplacian. To treat the nonlocal problem (2.1), we will study a corresponding extension problem, so that we can investigate problem (2.1) by studying a local problem via classical nonlinear variational methods.

We first define the extension operator and fractional Laplacian for functions in $H_0^s(\Omega)$.

Definition 3.2. Given a function $u \in H_0^s(\Omega)$, we define its s -harmonic extension $w = E_s(u)$ to the cylinder \mathcal{C} as a solution of the problem

$$\begin{cases} \operatorname{div}(y^{1-2s} \nabla w) = 0, & \text{in } \mathcal{C}, \\ w = 0, & \text{on } \partial_L \mathcal{C}, \\ w = u, & \text{on } \Omega \times \{0\}. \end{cases}$$

Following [1], we can define the fractional Laplacian operator by the Dirichlet to Neumann map as follows.

Definition 3.3. For any regular function $u(x)$, the fractional Laplacian $(-\Delta)^s$ acting on u is defined by

$$(-\Delta)^s u(x) = - \lim_{y \rightarrow 0^+} y^{1-2s} \frac{\partial w}{\partial y}(x, y), \quad \forall x \in \Omega, \quad y \in (0, \infty),$$

where $w = E_s(u)$.

From [1] and [12], the map $E_s(\cdot)$ is an isometry between $H_0^s(\Omega)$ and $H_{0,L}^s(\mathcal{C})$. Furthermore, we have

- (i) $\|(-\Delta)^s u\|_{H^{-s}(\Omega)} = \|u\|_{H_0^s(\Omega)} = \|E_s(u)\|_{H_{0,L}^s(\mathcal{C})}$, where $H^{-s}(\Omega)$ denotes the dual space of $H_0^s(\Omega)$;
- (ii) For any $w \in H_{0,L}^s(\mathcal{C})$, there exists a constant C independent of w such that

$$\|\operatorname{tr}_{\Omega} w\|_{L^r(\Omega)} \leq C \|w\|_{H_{0,L}^s(\mathcal{C})}$$

holds for every $r \in [2, \frac{2N}{N-2s}]$. Moreover, $H_{0,L}^s(\mathcal{C})$ is compactly embedded into $L^r(\Omega)$ for $r \in [2, \frac{2N}{N-2s})$.

In the following lemma, we will list some inequalities.

Lemma 3.4. *For every $1 \leq r \leq \frac{2N}{N-2s}$, and every $w \in H_{0,L}^s(\mathcal{C})$, it holds*

$$\left(\int_{\Omega \times \{0\}} |w|^r dx \right)^{\frac{2}{r}} \leq C \int_{\mathcal{C}} y^{1-2s} |\nabla w|^2 dx dy,$$

where constant C depends on $r, s, N, |\Omega|$.

Lemma 3.5. *For every $w \in H^s(\mathbb{R}_+^{N+1})$ the sharp fractional Sobolev inequality for $N > 2s$ and $s > 0$*

$$\left(\int_{\mathbb{R}^N} |u(x)|^{2^*} dx \right)^{\frac{2}{2^*}} \leq S \int_{\mathbb{R}_+^{N+1}} y^{1-2s} |\nabla w(x, y)|^2 dx dy,$$

which holds with the constant

$$S = \frac{2^{-1} \pi^{-s} \Gamma(s) \Gamma(\frac{N-2s}{2}) (\Gamma(N))^{\frac{2s}{N}}}{\Gamma(\frac{2-2s}{2}) \Gamma(\frac{N+2s}{2}) (\Gamma(\frac{N}{2}))^{\frac{2s}{N}}},$$

where $u = \text{tr}_\Omega w$.

Theorem 2.1 and 2.2 will be proved by using a recent result on the existence of at least three critical points by Ricceri [14, 15]. For the reader's convenience, we describe it as follows.

If X is a real Banach space, we can denote by \mathcal{X} the class of all function $\phi : X \rightarrow \mathbb{R}$ possessing the following property: if $\{u_n\} \subset X$ is a sequence converging weakly to $u \in X$ and $\liminf_{n \rightarrow \infty} \phi(u_n) \leq \phi(u)$, then $\{u_n\}$ has a subsequence converging strongly to u .

Theorem 3.6. *Let X be a separable and reflexive real Banach space; $I \subseteq \mathbb{R}$ an interval; $\Phi : X \rightarrow \mathbb{R}$ a sequentially weakly lower semi-continuous C^1 functional, belonging to \mathcal{X} , bounded on each bounded subset of X and whose derivative admits a continuous inverse on X^* ; $J : X \rightarrow \mathbb{R}$ a C^1 functional with compact derivative. Assume that, for each $\lambda \in I$, the functional $\Phi - \lambda J$ is coercive and has a strict local, not global minimum, say \hat{u}_λ . Then, for each compact interval $[a, b] \subseteq I$ for which $\sup_{\lambda \in [a, b]} (\Phi(\hat{u}_\lambda) - \lambda J(\hat{u}_\lambda)) < +\infty$, there exists $\gamma > 0$ with the following property: for every $\lambda \in [a, b]$ and every C^1 functional $\Psi : X \rightarrow \mathbb{R}$ with compact derivative, there exists $\delta_0 > 0$ such that, for each $\mu \in [0, \delta_0]$, the equation*

$$\Phi'(u) = \lambda J'(u) + \mu \Psi'(u)$$

has at least three solutions whose norm are less than γ .

4. PROOF OF THE MAIN RESULTS

Let

$$\Psi(u) = \frac{1}{2} \int_{\mathcal{C}} y^{1-2s} |\nabla w|^2 dx dy, \quad J(w) = \int_{\Omega \times \{0\}} F(x, w) dx.$$

Obviously, the condition (f_3) implies

$$\theta = \sup_{\Psi(w) \neq 0} \frac{J(w)}{\Psi(w)} > 0.$$

Lemma 4.1. *Let f satisfy $(f_1) - (f_3)$. Then for every $\lambda \in (0, \infty)$, the functional $\Phi - \lambda J$ is sequentially weakly lower continuous and coercive on $H_{0,L}^s(\mathcal{C})$, and has a global minimizer w_λ .*

Proof. By (f_1) and (f_3) , for any $\varepsilon > 0$, there exist $M_0 > 0$ and $C_1 > 0$ such that, for all $s \in H_{0,L}^s(\mathcal{C})$,

$$|f(x, s)| \leq \varepsilon |s|, \quad \forall |s| \geq M_0,$$

and

$$|f(x, s)| \leq C_1, \quad \forall |s| \leq M_0 + 1.$$

So, for any $w \in H_{0,L}^s(\mathcal{C})$, we have

$$|f(x, w)| \leq C_1 + \varepsilon |w|, \quad (4.1)$$

which implied that

$$|F(x, w)| \leq C_1 |w| + \frac{\varepsilon}{2} |w|^2.$$

Thus, for all $w \in H_{0,L}^s(\mathcal{C})$, we obtain

$$\begin{aligned} \Phi(w) - \lambda J(w) &= \frac{1}{2} \int_{\mathcal{C}} y^{1-2s} |\nabla w|^2 dx dy - \lambda \int_{\Omega \times \{0\}} F(x, w) dx \\ &\geq \frac{1}{2} \int_{\mathcal{C}} y^{1-2s} |\nabla w|^2 dx dy - \lambda \int_{\Omega \times \{0\}} (C_1 |w| + \frac{\varepsilon}{2} |w|^2) dx \\ &= \frac{1}{2} \|w\|_{H_{0,L}^s(\mathcal{C})}^2 - \lambda \frac{\varepsilon}{2} \int_{\Omega \times \{0\}} |w|^2 dx - \lambda C_1 \int_{\Omega \times \{0\}} |w| dx \\ &\geq \left(\frac{1}{2} - \lambda \frac{\varepsilon C_2}{2} \right) \|w\|_{H_{0,L}^s(\mathcal{C})}^2 - \lambda C_1 C_3 \|w\|_{H_{0,L}^s(\mathcal{C})}, \end{aligned}$$

where constants $C_2 > 0$, $C_3 > 0$. Let $\varepsilon > 0$ small enough such that $\frac{1}{2} - \lambda \frac{\varepsilon C_2}{2} > 0$, then we have

$$\Phi(w) - \lambda J(w) \rightarrow +\infty \quad \text{as} \quad \|w\|_{H_{0,L}^s(\mathcal{C})} \rightarrow \infty.$$

Hence, $\Phi - \lambda J$ is coercive.

Moreover, from the embedding $H_{0,L}^s(\mathcal{C}) \hookrightarrow L^r(\Omega)$ ($1 \leq r < 2_s^*$) is compact and (4.1), J is weakly continuous. Obviously,

$$\Phi(u) = \frac{1}{2} \int_{\mathcal{C}} y^{1-2s} |\nabla w|^2 dx dy = \frac{1}{2} \|w\|_{H_{0,L}^s(\mathcal{C})}^2$$

is weakly lower semi-continuous on $H_{0,L}^s(\mathcal{C})$. We can deduce that $\Phi - \lambda J$ is a sequentially weakly lower semi-continuous. So, $\Phi - \lambda J$ has a global minimizer $w_\lambda \in H_{0,L}^s(\mathcal{C})$. The proof is completed. \square

Next, we will show that $\Phi - \lambda J$ has a strictly local, not global minimizer for some λ , when f satisfies $(f_1) - (f_3)$.

Lemma 4.2. *Let f satisfy $(f_1) - (f_3)$. Then*

- (i) 0 is a strict local minimizer of the functional $\Phi - \lambda J$ for $\lambda \in (0, +\infty)$.
- (ii) $w_\lambda \neq 0$, i.e., 0 is not the global minimizer w_λ for $\lambda \in (\frac{1}{\theta}, +\infty)$, where w_λ is given by Lemma 4.1.

Proof. Firstly, we prove that

$$\lim_{\|w\|_{H_{0,L}^s(\mathcal{C})} \rightarrow 0} \frac{J(w)}{\Phi(w)} = 0, \quad \forall w \in H_{0,L}^s(\mathcal{C}).$$

In fact, by (f_2) , for any $\varepsilon > 0$, there exists a $\delta > 0$, such that

$$|f(x, w)| \leq \varepsilon |w|, \quad |w| < \delta. \quad (4.2)$$

Considering the inequality (4.2), (f_1) and (f_3) , there exists $r \in (1, 2_s^* - 1)$ such that

$$|f(x, w)| \leq \varepsilon |w| + |w|^r. \quad (4.3)$$

Then from Lemma 3.4, there exist $C_4, C_5 > 0$, such that

$$|J(w)| \leq \varepsilon C_4 \|w\|_{H_{0,L}^s(\mathcal{C})}^2 + C_5 \|w\|_{H_{0,L}^s(\mathcal{C})}^{r+1}.$$

This implies

$$\lim_{\|w\|_{H_{0,L}^s(\mathcal{C})} \rightarrow 0} \frac{J(w)}{\Phi(w)} = 0.$$

Next, we will prove (i) and (ii).

- (i) For $\lambda \in (0, +\infty)$, since $\lim_{\|w\|_{H_{0,L}^s(\mathcal{C})} \rightarrow 0} \frac{J(w)}{\Phi(w)} = 0 < \frac{1}{\lambda}$ and $\Phi(w) > 0$ for each $w \neq 0$ in some neighborhood U of 0, there exists a neighborhood $V \subseteq U$ of 0 such that

$$\Phi(w) - \lambda J(w) > 0, \quad \forall w \in V \setminus \{0\}.$$

Hence, 0 is a strict local minimum of $\Phi - \lambda J$.

- (ii) For $\lambda \in (\frac{1}{\theta}, +\infty)$, from the definition of θ , there exists $\hat{w} \in H_{0,L}^s(\mathcal{C})$ such that $\Phi(\hat{w}) > 0$, $J(\hat{w}) > 0$ and $\frac{J(\hat{w})}{\Phi(\hat{w})} > \frac{1}{\lambda}$. So we have

$$\Phi(\hat{w}) - \lambda J(\hat{w}) < 0 = \Phi(0) - \lambda J(0).$$

This yields 0 is not a global minimum of $\Phi - \lambda J$.

This completes the proof. \square

Let $K > 0$ be a real number, whose value will be fixed latter. Define the truncation of $|w|^{p-2}w$ with $p > 2_s^*$, be given by

$$g_K(w) = \begin{cases} |w|^{p-2}w, & \text{if } 0 \leq |w| \leq K, \\ K^{p-q}|w|^{q-2}w, & \text{if } |w| > K, \end{cases}$$

where $q \in (2, 2_s^*)$. Then $g_K(w)$ satisfies

$$|g_K(w)| \leq K^{p-q}|w|^{q-1},$$

for K large enough. Then, we study the truncated problem

$$\begin{cases} \operatorname{div}(y^{1-2s}\nabla w) = 0, & \text{in } \mathcal{C}, \\ w = 0, & \text{on } \partial_L \mathcal{C}, \\ \partial_\nu^s w = \lambda f(x, w) + \mu g_K(w), & \text{in } \Omega \times \{0\}, \end{cases} \quad (4.4)$$

We say that $w \in H_{0,L}^s(\mathcal{C})$ is a weak solution of the problem (4.4) if

$$\int_{\mathcal{C}} y^{1-2s} \nabla w \cdot \nabla \varphi dx dy = \lambda \int_{\Omega \times \{0\}} f(x, w) \varphi dx + \mu \int_{\Omega \times \{0\}} g_K(w) \varphi dx \quad (4.5)$$

for every $\varphi \in H_{0,L}^s(\mathcal{C})$.

Let

$$\Psi(u) = \int_{\Omega \times \{0\}} G_K(w) dx,$$

where $G_K(w) = \int_0^u g_K(t) dt$. So from $|g_K(w)| \leq K^{p-q}|w|^{q-1}$, $2 < q < 2_s^*$, we get that $g_K(w)$ is a super-linear function with subcritical growth, then $\Psi(u)$ has a compact derivative in $H_{0,L}^s(\mathcal{C})$. Moreover, for each compact interval $[a, b] \subset (\frac{1}{\theta}, +\infty)$, $\lambda \in [a, b]$. From (4.3), we have then $J(w)$ has a compact derivative in $H_{0,L}^s(\mathcal{C})$ too. Therefore, it is easy to see that the functional

$$\mathcal{E}(w) = \Phi(w) - \lambda J(w) - \mu \Psi(w) \quad \forall w \in H_{0,L}^s(\mathcal{C})$$

is C^1 and its derivative is given by

$$\langle \mathcal{E}'(w), \varphi \rangle = \int_{\mathcal{C}} y^{1-2s} \nabla w \nabla \varphi dx dy - \lambda \int_{\Omega \times \{0\}} f(x, w) \varphi dx - \mu \int_{\Omega \times \{0\}} g_K(w) \varphi dx,$$

for all $\varphi \in H_{0,L}^s(\mathcal{C})$.

By Lemma 4.1 and Lemma 4.2, all the hypotheses of Theorem 3.6 are satisfied. So there exists $\gamma > 0$ with the following property: for every $\lambda \in [a, b] \subset (\frac{1}{\theta}, +\infty)$, there exists $\delta_0 > 0$, such that for $\mu \in [0, \delta_0]$, the problem (4.4) has at least three solutions w_0 , w_1 and w_2 in $H_{0,L}^s(\mathcal{C})$ and

$$\|w_k\|_{H_{0,L}^s(\mathcal{C})} \leq \gamma, \quad k = 0, 1, 2,$$

where γ depends on λ , but does not depend on μ or K .

If the three solutions w_k , $k = 0, 1, 2$, satisfy

$$|w_k| \leq K, \quad \text{a.e. } (x, y) \in \Omega \times (0, \infty), \quad k = 0, 1, 2. \quad (4.6)$$

Then in the view of the definition g_K , we have $g_K(x, w) = \mu|w|^{p-2}w$ and therefore w_k , $k = 0, 1, 2$, are also solutions of the original problem (2.2). Thus, in order to prove Theorem 2.1, it suffices to show that exists $\delta_0 > 0$, such that for $\mu \in [0, \delta_0]$, the solutions obtained by Theorem 3.6 satisfy the inequality (4.6).

Proof of theorem 2.1. Our aim is to show that exists $\delta_0 > 0$, such that for $\mu \in [0, \delta_0]$, the solution w_k , $k = 0, 1, 2$, satisfy the inequality (4.6). To save notation, we will denote $w := w_k$, $k = 0, 1, 2$.

Set $w_+ = \max\{w, 0\}$, $w_- = -\min\{w, 0\}$. Then $|w| = w_+ + w_-$. We can argue with the positive and negation part of w separately.

We first deal with w_+ . For each $L > 0$, we define the following functions

$$w_L = \begin{cases} w_+, & \text{if } w_+ \leq L, \\ L, & \text{if } w_+ > L. \end{cases}$$

For $\beta > 1$ to be determined, we choose in (4.5) that

$$\varphi = w_L^{2(\beta-1)} w_+,$$

and since

$$\nabla \varphi = w_L^{2(\beta-1)} \nabla w_+ + 2(\beta-1) w_L^{2(\beta-1)-1} w_+ \nabla w_L,$$

we obtain

$$\begin{aligned}
& \int_{\mathcal{C}} y^{1-2s} \nabla w \nabla \varphi dx dy \\
&= \int_{\mathcal{C}} y^{1-2s} (\nabla(w_+ - w_-)) \nabla(w_L^{2(\beta-1)} w_+) dx dy \\
&= \int_{\mathcal{C}} y^{1-2s} (\nabla w_+ - \nabla w_-) (w_L^{2(\beta-1)} \nabla w_+ + 2(\beta-1) w_L^{2(\beta-1)-1} w_+ \nabla w_L) dx dy \\
&= \int_{\mathcal{C}} y^{1-2s} (|\nabla w_+|^2 w_L^{2(\beta-1)} + 2(\beta-1) w_L^{2(\beta-1)-1} w_+ \nabla w_L \nabla w_+) dx dy \\
&= \int_{\mathcal{C}} y^{1-2s} w_L^{2(\beta-1)} |\nabla w_+|^2 dx dy + 2(\beta-1) \int_{\mathcal{C}} y^{1-2s} w_L^{2(\beta-1)-1} w_+ \nabla w_L \nabla w_+ dx dy.
\end{aligned} \tag{4.7}$$

From the definition of w_L , we have

$$\begin{aligned}
& 2(\beta-1) \int_{\mathcal{C}} y^{1-2s} w_L^{2(\beta-1)-1} w_+ \nabla w_L \nabla w_+ dx dy \\
&= 2(\beta-1) \int_{\{w_+ < L\}} y^{1-2s} w_L^{2(\beta-1)-1} w_+ \nabla w_L \nabla w_+ dx dy \\
&= 2(\beta-1) \int_{\{w_+ < L\}} y^{1-2s} w_+^{2(\beta-1)} |\nabla w_+|^2 dx dy \\
&\geq 0.
\end{aligned} \tag{4.8}$$

Set

$$h_K(x, w) = \lambda f(x, w) + \mu g_K(x, w), \quad \forall w \in H_{0,L}^s(\mathcal{C}).$$

From (4.3) and $|g_K(x, w)| \leq K^{p-q} |w|^{q-1}$, we can choose constant $C_6 > 0$ such that

$$|h_K(x, w)| \leq C_6 |w| + \mu K^{p-q} |w|^{q-1}. \tag{4.9}$$

We deduce from (4.5), (4.7), (4.8) and (4.9) for $\beta > 1$ that

$$\begin{aligned}
\int_{\mathcal{C}} y^{1-2s} w_L^{2(\beta-1)} |\nabla w_+|^2 dx dy &= \int_{\Omega \times \{0\}} h_K(x, w) \varphi dx \leq \int_{\Omega \times \{0\}} |h_K(x, w) \varphi| dx \\
&\leq \int_{\Omega \times \{0\}} (C_6 |w| + \mu K^{p-q} |w|^{q-1}) w_L^{2(\beta-1)} w_+ dx \\
&= \int_{\Omega \times \{0\}} (C_6 (w_+ + w_-) + \mu K^{p-q} (w_+ + w_-)^{q-1}) w_L^{2(\beta-1)} w_+ dx \\
&= \int_{\Omega \times \{0\}} (C_6 w_+^2 w_L^{2(\beta-1)} + \mu K^{p-q} w_+^q w_L^{2(\beta-1)}) dx.
\end{aligned} \tag{4.10}$$

Let $\hat{w}_L = w_+ w_L^{\beta-1}$, we have

$$\nabla \hat{w}_L = w_L^{\beta-1} \nabla w_+ + (\beta-1) w_+ w_L^{\beta-2} \nabla w_L.$$

By the Sobolev embedding theorem,

$$\begin{aligned}
& \left(\int_{\Omega \times \{0\}} |\hat{w}_L|^{2_s^*} dx \right)^{\frac{2}{2_s^*}} \leq S \int_{\mathcal{C}} y^{1-2s} |\nabla \hat{w}_L|^2 dx dy \\
& = C \int_{\mathcal{C}} y^{1-2s} |w_L^{\beta-1} \nabla w_+ + (\beta-1) w_+ w_L^{\beta-2} \nabla w_L|^2 dx dy \\
& \leq 2S \left(\int_{\mathcal{C}} y^{1-2s} |(\beta-1) w_+ w_L^{\beta-2} \nabla w_L|^2 dx dy + \int_{\mathcal{C}} y^{1-2s} |w_L^{\beta-1} \nabla w_+|^2 dx dy \right) \\
& = 2S \left(\int_{\mathcal{C}} y^{1-2s} (\beta-1)^2 |w_L|^{2(\beta-1)} |\nabla w_+|^2 dx dy + \int_{\mathcal{C}} y^{1-2s} |w_L^{\beta-1} \nabla w_+|^2 dx dy \right) \\
& \leq 2S \left((\beta-1)^2 \int_{\mathcal{C}} y^{1-2s} w_L^{2(\beta-1)} |\nabla w_+|^2 dx dy + \int_{\mathcal{C}} y^{1-2s} w_L^{2(\beta-1)} |\nabla w_+|^2 dx dy \right) \\
& = 2S \left((\beta-1)^2 + 1 \right) \int_{\mathcal{C}} y^{1-2s} w_L^{2(\beta-1)} |\nabla w_+|^2 dx dy \\
& = 2S \beta^2 \left(\left(\frac{\beta-1}{\beta} \right)^2 + \frac{1}{\beta^2} \right) \int_{\mathcal{C}} y^{1-2s} w_L^{2(\beta-1)} |\nabla w_+|^2 dx dy,
\end{aligned} \tag{4.11}$$

where $S > 0$ is the Sobolev embedding constant.

Since $\beta > 1$, we have $\frac{1}{\beta^2} < 1$ and $\left(\frac{\beta-1}{\beta} \right)^2 < 1$. From (4.10) and (4.11), we get

$$\begin{aligned}
& 2S \beta^2 \left(\left(\frac{\beta-1}{\beta} \right)^2 + \frac{1}{\beta^2} \right) \int_{\mathcal{C}} y^{1-2s} w_L^{2(\beta-1)} |\nabla w_+|^2 dx dy \\
& < 4S \beta^2 \int_{\mathcal{C}} y^{1-2s} w_L^{2(\beta-1)} |\nabla w_+|^2 dx dy \\
& \leq 4S \beta^2 \int_{\Omega \times \{0\}} (C_6 w_+^2 w_L^{2(\beta-1)} + \mu K^{p-q} w_+^q w_L^{2(\beta-1)}) dx.
\end{aligned} \tag{4.12}$$

From the Sobolev embedding $H_{0,L}^s(\mathcal{C}) \hookrightarrow L^{2_s^*}(\Omega)$ and $\|w_+\|_{H_{0,L}^s(\mathcal{C})} \leq \gamma$, we have

$$\left(\int_{\Omega \times \{0\}} |w_+|^{2_s^*} dx \right)^{\frac{2}{2_s^*}} \leq S \int_{\mathcal{C}} y^{1-2s} |\nabla w_+|^2 dx dy \leq S \gamma. \tag{4.13}$$

Let $t = \frac{2s^*}{2s^*-q+2}$. Since $w_+^q w_L^{2(\beta-1)} = w_+^q \hat{w}_L^2 w_+^{-2} = w_+^{q-2} \hat{w}_L^2$ and $\hat{w}_L^2 = w_+^2 w_L^{2(\beta-1)}$, we can use the Hölder's inequality, (4.11), (4.12) and (4.13) to conclude that, whenever $\hat{w}_L(\cdot, 0) \in L^t(\Omega)$, it holds

$$\begin{aligned}
& \left(\int_{\Omega \times \{0\}} |\hat{w}_L|^{2s^*} dx \right)^{\frac{2}{s^*}} \\
& \leq 4S \beta^2 \int_{\Omega \times \{0\}} \left(C w_+^2 w_L^{2(\beta-1)} + \mu K^{p-q} w_+^q w_L^{2(\beta-1)} \right) dx, \\
& = 4S \beta^2 \left(C \int_{\Omega \times \{0\}} \hat{w}_L^2 dx + \mu K^{p-q} \int_{\Omega \times \{0\}} w_+^{q-2} \hat{w}_L^2 dx \right), \\
& \leq 4S \beta^2 \left[|\Omega|^{\frac{q-2}{2s^*}} \left(\int_{\Omega \times \{0\}} \hat{w}_L^t dx \right)^{\frac{2}{t}} + \mu K^{p-q} \left(\int_{\Omega \times \{0\}} |w_+|^{2s^*} dx \right)^{\frac{q-2}{2s^*}} \left(\int_{\Omega \times \{0\}} \hat{w}_L^t dx \right)^{\frac{2}{t}} \right] \\
& \leq 4S \beta^2 \left(|\Omega|^{\frac{q-2}{2s^*}} + \mu K^{p-q} (S \gamma)^{\frac{q-2}{2}} \right) \|\hat{w}_L\|_{L^t}^2.
\end{aligned}$$

Set $\beta := \frac{2s^*}{t} = 1 + \frac{2s^*-q}{2} > 1$. By the definition of w_L , we have $w_L \leq w_+$, then we conclude that $\hat{w}_L(\cdot, 0) \in L^t(\Omega)$, whenever $(w_+(\cdot, 0))^\beta \in L^t(\Omega)$. If this is the case, it follow from the above inequality that

$$\begin{aligned}
\left(\int_{\Omega \times \{0\}} |\hat{w}_L|^{2s^*} dx \right)^{\frac{2}{s^*}} &= \left(\int_{\Omega \times \{0\}} w_L^{2s^*(\beta-1)} w_+^{2s^*} dx \right)^{\frac{2}{s^*}} \\
&\leq 4S \beta^2 \left(|\Omega|^{\frac{q-2}{2s^*}} + \mu K^{p-q} (S \gamma)^{\frac{q-2}{2}} \right) \left(\int_{\Omega \times \{0\}} |w_L^{\beta-1} w_+|^t dx \right)^{\frac{2}{t}}.
\end{aligned}$$

By Fatou's Lemma in the variable L , we get

$$\left(\int_{\Omega \times \{0\}} w_+^{2s^* \beta} dx \right)^{\frac{2\beta}{2s^* \beta}} \leq 4S \beta^2 C_{\mu, K} \left(\int_{\Omega \times \{0\}} |w_+|^t dx \right)^{\frac{2\beta}{t\beta}},$$

i.e.,

$$\left(\int_{\Omega \times \{0\}} w_+^{2s^* \beta} dx \right)^{\frac{1}{2s^* \beta}} \leq \left(4S C_{\mu, K} \right)^{\frac{1}{2\beta}} \beta^{\frac{1}{\beta}} \left(\int_{\Omega \times \{0\}} |w_+|^t dx \right)^{\frac{1}{t\beta}}, \quad (4.14)$$

where $C_{\mu, K} = |\Omega|^{\frac{q-2}{2s^*}} + \mu K^{p-q} (S \gamma)^{\frac{q-2}{2}}$.

Since $\beta = \frac{2s^*}{t} > 1$ and $w_+(\cdot, 0) \in L^{2s^*}(\Omega)$, the inequality (4.14) holds for this choice of β . Therefore, from $\beta^2 t = \beta 2s^*$, we have that the inequality (4.14) also

holds with β replaced by β^2 . Hence

$$\begin{aligned}
\left(\int_{\Omega \times \{0\}} w_+^{2_s^* \beta^2} dx \right)^{\frac{1}{2_s^* \beta^2}} &\leq \left(4S C_{\mu, K} \right)^{\frac{1}{2\beta^2}} (\beta^2)^{\frac{1}{\beta^2}} \left(\int_{\Omega \times \{0\}} |w_+|^{t\beta^2} dx \right)^{\frac{1}{t\beta^2}} \\
&= \left(4S C_{\mu, K} \right)^{\frac{1}{2\beta^2}} (\beta^2)^{\frac{1}{\beta^2}} \left(\int_{\Omega \times \{0\}} |w_+|^{2_s^* \beta} dx \right)^{\frac{1}{2_s^* \beta}} \\
&\leq \left(4S C_{\mu, K} \right)^{\frac{1}{2\beta^2}} (\beta^2)^{\frac{1}{\beta^2}} \left(4S C_{\mu, K} \right)^{\frac{1}{2\beta}} \beta^{\frac{1}{\beta}} \left(\int_{\Omega \times \{0\}} |w_+|^{t\beta} dx \right)^{\frac{1}{t\beta}} \\
&= \left(4S C_{\mu, K} \right)^{\frac{1}{2}(\frac{1}{\beta^2} + \frac{1}{\beta})} \beta^{\frac{2}{\beta^2} + \frac{1}{\beta}} \left(\int_{\Omega \times \{0\}} |w_+|^{t\beta} dx \right)^{\frac{1}{t\beta}}
\end{aligned}$$

By iterating this process and $\beta t = 2_s^*$, we obtain

$$\left(\int_{\Omega \times \{0\}} w_+^{2_s^* \beta^m} dx \right)^{\frac{1}{2_s^* \beta^m}} \leq \left(4S C_{\mu, K} \right)^{\frac{1}{2}(\frac{1}{\beta^m} + \dots + \frac{1}{\beta^2} + \frac{1}{\beta})} \beta^{\frac{m}{\beta^m} + \dots + \frac{2}{\beta^2} + \frac{1}{\beta}} \left(\int_{\Omega \times \{0\}} |w_+|^{2_s^*} dx \right)^{\frac{1}{2_s^*}}. \quad (4.15)$$

Taking the limit as $m \rightarrow \infty$ in (4.15), we have

$$\|w_+\|_{L^\infty} \leq (4S C_{\mu, K})^{\theta_1} \beta^{\theta_2} \|w_+\|_{L^{2_s^*}} \leq (4S C_{\mu, K})^{\theta_1} \beta^{\theta_2} (S \gamma)^{\frac{1}{2}},$$

where $\theta_1 = \frac{1}{2} \sum_{m=1}^{\infty} \frac{1}{\beta^m}$, $\theta_2 = \sum_{m=1}^{\infty} \frac{m}{\beta^m}$ and $\beta > 1$.

Next, we will find some suitable value of K and μ , such that the inequality

$$(4S C_{\mu, K})^{\theta_1} \beta^{\theta_2} (S \gamma)^{\frac{1}{2}} \leq \frac{K}{2} \quad (4.16)$$

holds. From (4.16), we get

$$C_{\mu, K} = |\Omega|^{\frac{q-2}{2_s^*}} + \mu K^{p-q} (S \gamma)^{\frac{q-2}{2}} \leq \frac{1}{4S} \left(\frac{K}{2(\gamma S)^{\frac{1}{2}} \beta^{\theta_2}} \right)^{\frac{1}{\theta_1}}.$$

Then, choose K to satisfy the inequality

$$\frac{1}{4S} \left(\frac{K}{2(\gamma S)^{\frac{1}{2}} \beta^{\theta_2}} \right)^{\frac{1}{\theta_1}} - |\Omega|^{\frac{q-2}{2_s^*}} > 0,$$

and fix μ_0 such that

$$0 < \mu_0 < \mu' := \frac{1}{K^{p-q} (S \gamma)^{\frac{q-2}{2}}} \left\{ \frac{1}{4S} \left(\frac{K}{2(\gamma S)^{\frac{1}{2}} \beta^{\theta_2}} \right)^{\frac{1}{\theta_1}} - |\Omega|^{\frac{q-2}{2_s^*}} \right\}.$$

Thus, we obtain (4.16) for $\mu \in [0, \mu_0]$, i.e.,

$$\|w_+\|_{L^\infty} \leq \frac{K}{2}, \quad \text{for } \mu \in [0, \mu_0]. \quad (4.17)$$

Similarly, we can also have the estimate for the w_- , i.e.,

$$\|w_-\|_{L^\infty} \leq \frac{K}{2}, \quad \text{for } \mu \in [0, \mu_0]. \quad (4.18)$$

Now, let $\delta = \min\{\delta_0, \mu_0\}$. For each $\mu \in [0, \delta]$, from (4.17), (4.18) and $|w| = w_+ + w_-$, we have

$$\|w\|_{L^\infty} \leq K, \quad \text{for } \mu \in [0, \mu_0].$$

Considering this fact and $w := w_k$, $k = 1, 2, 3$ we get

$$\|w_k\|_{L^\infty} \leq K, \quad k = 0, 1, 2, \quad \text{for } \mu \in [0, \delta].$$

Therefore, we obtain the inequality (4.6). The proof is completed.

Proof of theorem 2.2. In fact, the truncation of $g_K(x, s)$ can be given by

$$g_K(x, s) = \begin{cases} g(x, s) & \text{if } |s| \leq K \\ \min\{g(x, s), C_0(1 + K^{p-q}|s|^{q-2}s)\} & \text{if } |s| > K \end{cases}$$

where $q \in (2, 2_s^*)$, Then g_K satisfies

$$|g_K(x, s)| \leq C_0(1 + K^{p-q}|s|^{q-2}), \quad \forall s \in \mathbb{R}.$$

Let $h_K(x, w) = \lambda f(x, w) + \mu g_K(x, w)$, $\forall w \in H_{0,L}^s(\mathcal{C})$. The truncated problems associated to h_K

$$\begin{cases} \operatorname{div}(y^{1-2s}\nabla w) = 0, & \text{in } \mathcal{C}, \\ w = 0, & \text{on } \partial_L \mathcal{C}, \\ \partial_\nu^s w = h_K(x, w), & \text{in } \Omega \times \{0\}. \end{cases} \quad (4.19)$$

Similar the proof of the Theorem 2.1, using Theorem 3.6 we can prove that there exists $\delta > 0$ such that the solutions w for the truncated problems (4.19) satisfy $\|w\|_{L^\infty} \leq K$ for $\mu \in [0, \delta]$; and in view of the definition g_K , we have

$$h_K(x, w) = \lambda f(x, w) + \mu g(x, w).$$

Therefore $w := w_k$, $k = 0, 1, 2$, are also solutions of the original problem (2.3).

REFERENCES

- [1] L. Caffarelli and L. Silvestre, An extension problem related to the fractional Laplacian, *Comm. Partial Differential Equations* 32(2007)1245–1260.
- [2] X. Cabré and J. Tan, Positive solutions for nonlinear problems involving the square root of the Laplacian, *Adv. Math.* 224 (2010) 2052–2093.
- [3] J. Tan, The Brezis-Nirenberg type problem involving the square root of the Laplacian, *Calc. Var.* 42 (2011)21–41.
- [4] W. Choi, S. Kim and K. Lee, Asymptotic behavior of solutions for nonlinear elliptic problems with the fractional Laplacian, *arXiv:1308.4206v1*.
- [5] Z.-C. Han, Asymptotic approach to singular solutions for nonlinear elliptic equations involving critical Sobolev exponent, *Ann. Inst. H. Poincaré Anal. Non Linéaire* 8 (1991) 159–174.
- [6] O. Rey, The role of the Green's function in a nonlinear elliptic equation involving the critical SObolev exponent, *J. Funct. Anal.* 89 (1990) 1-52.
- [7] J. Cohabrowski, J. Yang, Existence theorems for elliptic equations involving supercritical Sobolev exponent, *Adv. Differential Equations* 2 (1997)231–256.
- [8] A. Ambrosetti, H. Brezis and G. Cerami, Combined effects of concave and convex nonlinearities in some elliptic problems, *J. Functional Analysis* 122 (1994)519–543.
- [9] J. Moser, A new proof of De Giorgi's theorem concerning the regularity problem for elliptic differential equations, *Comm. Pure Appl. Math.* 13 (1960)457–468.
- [10] J. Francisco, S. A. Correa, Giovany M. Figueiredo, On an elliptic equation of p-Kirchhoff type via variational methods, *Bull. Australian Math. Soc.* 74 (2006)263–277.
- [11] G.M. Figueiredo, M.F. Furtado, Positive solutions for some quasilinear equations with critical and supercritical growth, *Nonlinear Anal. TMA* 66(7)(2007)1600–1616.
- [12] B. Brändle, E. Colorado, A. de pablo and U. Sánchez, A concave-convex elliptic problem involving the fractional Laplacian, *Proc. Roy. Soc. Edinburgh. Sect. A* 143 (2013) 39–71.
- [13] L. Zhao, P. Zhao, The existence of solutions for p -Laplacian problems with critical and supercritical growth, To appear in *Rocky Mountain J. Math.*.
- [14] B. Ricceri, A three points theorem revisited, *Nonlinear Anal.* 70 (2009)3084–3089.
- [15] B. Ricceri, A further three points theorem, *Nonlinear Anal.* 71 (2009)4151–4157.

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